A GEOMETRIC APPROACH TO BURGERS'
RICCATI EQUATION

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THESIS

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ABSTRACT

The directrix approach, which gives powerful insight to the geometric structure of solutions of the general Riccati equation, is developed.

Burgers' Riccati equation is derived, and conclusions are drawn utilizing the directrix approach concerning the boundedness properties of solutions of this equation with certain restrictions on parameter values.

Closed form solutions are developed for Burgers' Riccati equation for certain parameter values. A method is produced to obtain the vertical assymptote for unbounded solutions of Burgers' Riccati equation with certain restrictions on parameter values. Conclusions drawn from the application of the directrix method to Burgers' equation are verified.

A research bibliography for Riccati's Equation is included.

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I. INTRODUCTION TO BURGERS' RICCATI EQUATION

A. PHYSICAL SETTING

The Riccati non-linear differential equation has many applications in mathematical physics. In this thesis a relatively unknown application of the Riccati equation to turbulence is explored.

Consider the equation

$$\frac{\partial v}{\partial t} + \mu v \frac{\partial v}{\partial y} - \frac{U}{b} v = \nu \frac{\partial^2 v}{\partial y^2} \qquad (1)$$

where v = v(y,t), U = U(t) and μ and ν are parameters. Equation (1) with μ = 2 appeared in J. M. Burgers' theory of a model of turbulence (See [1]). In this model U(t) is the analogue of the mean motion of a fluid through a channel. The independent variable y represents the cross-channel coordinate and t represents time. If the domain of y is taken to be [o,b], then v(y,t) represents the secondary motion or turbulence of the fluid moving through the channel under some external force.

If in Equation (1) U(t) $\equiv$ 0, then Equation (1) becomes

$$\frac{\partial v}{\partial t} + \mu v \frac{\partial v}{\partial y} = \nu \frac{\partial^2 v}{\partial y^2} \qquad (2)$$

Suppose that $y \in [0,b]$, v(y,0) is some known function vo(y), and v(0,t) = v(b,t) = 0. In 1951 J. Cole (see [2]) published the general solution of this system in terms of the heat equation

$$\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial y^2} ,$$

where

$$v(y,t) = -2\nu \frac{1}{\theta} \frac{\partial \theta}{\partial y} .$$

With regard to this system, where y takes on values in a finite range, the theory of (2) becomes completely understood, since the theory of the heat equation is completely understood.

If the eigenvalue-type restriction of the initial and final values of $v(y,t)$ are lifted, and if y is allowed to take on values in an infinite range, then $v(y,t)$ represents the free turbulence in an infinite channel. Thus, the problem that Cole solved is actually a special case of the free turbulence problem. It is possible that an extension or generalization of Cole's method of solution might lead to a complete solution of the free turbulence problem associated with equation (2).

However, Equation (2) will be approached from another direction - via the geometry of the classical form of the Riccati equation. The approach is of such breadth as to apply to Riccati equations of greater generality than those associated with (2).

To see how the connection with (2) and the Riccati equation is made, observe that in Ref. 1 a typical solution of (2) is put into the form

$$v = \frac{V(\eta)}{(t-t_0)^{\frac{1}{2}}} \quad ,$$

where

$$\eta = \frac{y-y_0}{(t-t_0)^{\frac{1}{2}}} \quad .$$

If the above transformation is applied to (2), a Riccati equation can be formed.

B. MATHEMATICAL DERIVATION

Thus, consider the following equation:

$$v_t + \mu v v_y = \nu v_{yy} \tag{1}$$

with parameters μ, ν. Let $v(\eta) = V(\eta)/(t - to)^{\frac{1}{2}}$ where $\eta = (y - yo)/(t - to)^{\frac{1}{2}}$. Then, substituting above and using

$$v_t = -(\eta V' + V)/2(t-t_0)^{\frac{3}{2}}$$

$$v_y = V'/(t-t_0)$$

and
$$v_{yy} = V''/(t-t_0)^{\frac{3}{2}}$$

where $'$ denotes $\frac{d}{d\eta}$,

(1) becomes
$$-\frac{\eta}{2} V' - \frac{V}{2} + \mu V V' = \nu V'' .$$

Thus,
$$\left[\nu V' - \frac{\mu}{2} V^2 + \frac{\eta}{2} V\right]' = 0 ,$$

and
$$\nu V' = \frac{\mu}{2} V^2 - \frac{\eta}{2} V + C^2$$

where C^2 is an integration constant.

In
$$\nu V' = \frac{\mu}{2}\left(V^2 - \frac{\eta}{\mu} V + \frac{2}{\mu} C^2\right)$$

let
$$Y = \frac{\mu\eta}{2\nu} .$$

Then
$$\nu V' = \nu \frac{dV}{dY}\frac{\mu}{2\nu} = \frac{\mu}{2}\left(V^2 - \frac{\eta}{\mu} V + \frac{2}{\mu} C^2\right),$$

or
$$\frac{dV}{dY} = V^2 - \frac{\eta}{\mu} V + \frac{2}{\mu} C^2 .$$

Now, since
$$\frac{\eta}{\mu} = Y \frac{2\nu}{\mu^2} ,$$

$$\frac{dV}{dY} = V^2 - 2\frac{\nu}{\mu^2} YV + \frac{2}{\mu} C^2 . \qquad (4)$$

Equation (4) will be referred to as Burgers' Riccati equation, or simply Burgers' equation. A more streamlined form in which Burgers' equation

will be handled is

$$y' = y^2 - 2\alpha x y + \beta \quad ,$$

where α and β are constants, and $'$ denotes $\frac{d}{dx}$.

II. DEVELOPMENT OF DIRECTRIX APPROACH

A. DEFINITION OF DIRECTRICES

Consider the general Riccati equation

$$y' = f(x) + g(x)y + h(x)y^2 \tag{5}$$

where f(x), g(x), and h(x) are real-valued functions of x. For purposes of this thesis, h(x) is assumed to be non-zero for all x. Since the right side of (5) is quadratic in y, (5) can be factored into the equivalent form:

$$y' = [y - u(x)][y - \ell(x)] \tag{6}$$

where $$u(x) = [-g(x) + (g^2(x) - 4f(x)g(x))^{\frac{1}{2}}]/2h(x)$$

and $$\ell(x) = [-g(x) - (g^2(x) - 4f(x)g(x))^{\frac{1}{2}}]/2h(x).$$

Let D be the set of all x such that u(x) and $\ell(x)$ are real-valued. For $x_o \epsilon D$, and y(x) a solution of (6), the value of $y'(x_o)$ depends upon the position of $y(x_o)$ relative to $u(x_o)$ and $\ell(x_o)$ (See Figure 1.). Clearly $y'(x_o) = 0$ if and only if either $y(x_o) = u(x_o)$ or $y(x_o) = \ell(x_o)$. Also, $y'(x_o) < 0$ if and only if $\ell(x_o) < y(x_o) < u(x_o)$, and $y'(x_o) > 0$ if and only if either $y(x_o) > u(x_o)$ or $y(x_o) < \ell(x_o)$. Knowing u(x) and $\ell(x)$, then, the solution may be "directed" at $x_o \epsilon D$, in the sense that the slope is known at x_o. Thus, u(x) and $\ell(x)$ "direct" the solution of (6) and will be referred to throughout this thesis as upper and lower directrices, respectively.

B. EFFECTS OF DIRECTRICES ON SOLUTIONS

From an intuitive point of view $u(x)$ appears to repel a solution $y(x)$ in the sense that for $x_o \in D$, $y(x_o) > u(x_o)$ implies that $y'(x_o) > 0$, and $\ell(x_o) < y(x_o) < u(x_o)$ implies that $y'(x_o) < 0$. Similarly, $\ell(x)$ appears to attract a solution $y(x)$ in the sense that $y(x_o) < \ell(x_o)$ implies that $y'(x_o) > 0$, and, once again, $\ell(x_o) < y(x_o) < u(x_o)$ implies that $y'(x_o) < 0$. A simple and interesting case of the repelling property of $u(x)$ and the attracting property of $\ell(x)$ occurs when $u(x) \equiv C_2 > C_1 \equiv \ell(x)$.

Suppose now that $u(x) \equiv C_2 > C_1 \equiv \ell(x)$, and that $y(x)$ is a solution of (6). It is interesting to observe that $y_2(x) \equiv C_2$ and $y_1(x) \equiv C_1$ are both solutions of (6). By a uniqueness argument, then, the only solution that can assume the value C_2 is $y_2(x)$, and similarly for C_1 and $y_1(x)$. Thus, the solutions of (6) with $u(x) \equiv C_2 > C_1 \equiv \ell(x)$ are partitioned into three distinct categories (not counting $y_1(x)$ and $y_2(x)$).

Suppose that $y(x)$ takes on values greater than C_2. Since $y(x) > C_2$ implies that $y'(x) > 0$, as $x \to \infty$, $y(x)$ is repelled from $u(x)$. In fact, since

$$y'' = 2\left[y'\left(y - \frac{C_1 + C_2}{2}\right)\right],$$

$y(x) > C_2$ implies that $y''(x) > 0$, so that $y(x)$ is concave upwards. Suppose that $\ell(x) < y(x) < u(x)$. Then for $y(x) > (C_1 + C_2)/2$, $y'(x) < 0$ and $y''(x) < 0$. Thus, $y(x)$ is repelled from $u(x)$. For $y < (C_1 + C_2)/2$, $y'(x) < 0$ and $y''(x) > 0$, so that $y(x)$ is assymptotic to $\ell(x) \equiv C_1$ as $x \to \infty$. That is, $y(x)$ is attracted by $\ell(x)$. Suppose finally that $y(x) < C_1$. Then $y'(x) > 0$ and $y''(x) > 0$ so that $y(x)$ is assymptotic to $\ell(x)$ as $x \to \infty$. Thus, in the case where $u(x) \equiv C_2 > C_1 \equiv \ell(x)$, solutions that are attracted to $\ell(x)$ are bounded as $x \to \infty$ and those solutions above $u(x)$ are repelled and become infinite as $x \to \infty$. (See Figure 2).

If the directrices are allowed to vary with x, the bounded and unbounded solutions of (6) can still be studied but are not as easily categorized as in the constant-directrix case. Specifically, if the directrices are such that solutions may intersect the directrices, perhaps even infinitely many times, the regions in which solutions become unbounded are relatively more difficult to ascertain. Nevertheless, observations at specific values of x can always be made concerning the "direction" of the solution, and deep intuitive insight into the forms of solutions is gained using the geometric structure of the upper and lower directrices.

C. APPLICATIONS

1. Riccati "Lens"

As an interesting and novel application of the directrix approach, consider the problem of constructing directrices in order that the solution have certain properties. For example, it has already been shown that solutions can be repelled from or attracted to a certain value K_1. In the first case, let $u(x) = K_1$ and $\ell(x) = K_2'$ where $K_1 > K_2'$. In the second case, let $\ell(x) = K_1$ and $u(x) = K_2$ where $K_1 < K_2$. The degree of attraction or replusion can be adjusted by varying K_2 or K_2'. Thus, a sort of Riccati "lens" can be constructed, in that the solutions are either focused (attracted) or scattered (repelled). (See Figure 3).

2. Oscillating Solutions

Suppose now that an osciallating solution is desired. Consider $\ell(x) = A\sin Bx + C$ and $u(x) \equiv K > A + C$. Let $y(x)$ be a solution of (6) and suppose that for some x_o, $\ell(x_o) < y(x_o) < u(x_o)$. Then

$y'(x_o) < 0$. As x increases from x_o, y(x) must eventually intersect $\ell(x)$, since $y'(x) < 0$ in this region. Since y(x) must intersect $\ell(x)$ at some point x_1, where $y'(x_1) = 0$, $\ell'(x_1)$ must be positive. As x increases from x_1, $y'(x) > 0$, so that y(x) must necessarily intersect $\ell(x)$ again, say at x_2. But this puts y(x) for $x > x_2$ between u(x) and $\ell(x)$ and the argument repeats. Furthermore, y(x) cannot assume a constant value for $x > \hat{x}$ since $y'(x) = 0$ implies that $y(x) \equiv K \equiv u(x)$ for $x > \hat{x}$. Thus, any solution $y(x) < u(x)$ oscillates for $x > x_o$. (See Figure 4).

3. Random Directrices

Another interesting problem is obtained by partitioning the x-axis into subintervals u_i, for i = 1, 2, 3,..., and allowing u(x) and $\ell(x)$ to be defined for each subinterval. Let u(x) and $\ell(x)$ be assigned random constant values for each subinterval. (See Figure 5). The figure then shows but one realization of an ensemble of u and ℓ segments. These u and ℓ values can have joint probability density functions of arbitrary type. Some interesting questions are:

(a) Given an initial value for a solution y(x) of (2), what is the probability that y(x) will be unbounded as $x \to \infty$?

(b) How does the answer to (a) depend upon the values that u(x) and $\ell(x)$ can assume?

4. Approximation of Continuous Directrices

Consider, finally, two real functions $f_1(x)$ and $f_2(x)$. Let the x-axis be partitioned as before, and define step function approximations of $f_1(x)$ and $f_2(x)$ on this partition. Let u(x) and $\ell(x)$ be defined for each interval as the value of the step function approximations to $f_1(x)$ and $f_2(x)$, respectively. Note that given an initial value y_o,

we can trace, through each subinterval, the solution of the Riccati equation defined by the values of u(x) and $\ell(x)$ on that subinterval. Suppose that the size of the subintervals approaches zero. Will this interval solution described above approach the solution through y_o of the Riccati equation defined by $f_1(x)$ and $f_2(x)$?

The consideration of all of these questions is beyond the scope of this thesis. They have been introduced to show the suggestive power of the directrix approach to the Riccati equation.

5. "Inverse" Problem

It is widely known of the Riccati equation that a general solution can be obtained from a non-trivial particular solution by means of two quadratures. Thus, the problem of finding general solutions reduces to that of finding non-trivial particular solutions. In some cases a simple regrouping of the terms in a Riccati equation can lead to a particular solution by inspection. Sugai [Ref. 3], for example, gives some particular solutions for cases in which the coefficients in Riccati's equation satisfy a prescribed interrelationship. The factoring of Riccati's equation, however, lends further insight to the problem of finding particular solutions. Suppose that f(x) is a real-valued function. A Riccati equation that is satisfied by f(x) can be found by making certain assumptions about the directrices of the Riccati equation. Specifically, if one directrix is required to be a certain function, then the other directrix can be found through a computation involving f(x) and the known directrix. Then, since both directrices are specified, the Riccati equation satisfied by f(x) is completely specified.

The directrix approach, then, suggests the following "inverse" problem. Given one directrix, $d_1(x)$, and a function f(x), what must $d_2(x)$

be in order that f(x) satisfy the Riccati equation $y' = (y-d_1)(y-d_2)$? Here the notation u(x) and $\ell(x)$ for directrices has been changed to $d_1(x)$ and $d_2(x)$ since the "upper" and "lower" ordering may be lost during crossings, i.e., u(x) (or $\ell(x)$) may turn out to be defined as $d_1(x)$ in some region and as $d_2(x)$ in another region. Solving for d_2,

$$d_2 = \frac{f' - f^2 + d_1 f}{d_1 - f} \quad . \tag{7}$$

Consider, for example, the Riccati equation

$$y' = y(y - d_2) \tag{8}$$

That is, $d_1 = 0$ and, from (7),

$$d_2 = f - f'/f$$

Suppose, for example, that f(x) is a polynomial of degree $n > 0$ such that $f(x) \neq 0$ when $x \in D'$ for some domain D'. What must $d_2(x)$ be in order that f(x) satisfy a Riccati equation of type (8)? From (8) it is clear that $\deg(f-d_2) = \frac{n-1}{n}$. Since $\frac{n-1}{n}$ must be an integer, $n = 1$. Hence, $f - d_2 = C$. But $f - d_2 = C$ implies that $f' = Cf$; that is, $f = \exp(Cx)$. Thus (8) has no polynomial solutions of degree $n > 0$.

This example serves actually two purposes. The first purpose, of course, is to point out that, using this method to pair particular solutions to equations, not every function can be realized as a particular solution of a Riccati equation of specified form. The second purpose is that the example leads into the problem of finding polynomial particular solutions of a Riccati equation whose coefficients are polynomials. This problem was solved in 1954 by Campbell and Goulomb [Ref. 4], for the Riccati equation

$$Ay' = B_0 + B_1 y + B_2 y^2 \tag{9}$$

where A, B_o, B_1, B_2 are polynomials in x. In Ref. 4 criteria are presented for the existence of, and an algorithm is presented for finding all polynomial solutions of (9). For the solution to the problem for $A \equiv B \equiv 1$, see Rainville [Ref. 5].

6. Tables of Particular Solutions

Suppose, now, that f(x) = a/x and $d_1 = 0$. From (7), $d_2 = (1+a)/x$ so that f(x) satisfies

$$y' = y\left(y - \left(\frac{1+a}{x}\right)\right)$$

Using this procedure, with further choices of f(x) and $d_1(x)$, a table of specific Riccati equations and the corresponding particular solutions could be constructed. Thus, the directrix approach provides a systematic procedure for constructing the Riccati equations associated with particular choices of one of the directrices of the equation and an arbitrary solution.

D. EXAMPLE

1. Template Solution

Suppose, on the other hand, that a specific Riccati equation is given, and that the solutions are to be constructed. The directrix approach yields insight to the behavior of solutions in certain regions of the plane. It is instructive to consider the example that follows.

Consider the Riccati equation

$$y' = y^2 + by + c \tag{10}$$

where b and c are real constants.

An equivalent form of (10) is

$$y' = (y-r_1)(y-r_2)$$

where

$$r_1 = [-b + (b^2 - 4c)^{\frac{1}{2}}]/2 \quad ,$$

and

$$r_2 = [-b - (b^2 - 4c)^{\frac{1}{2}}]/2 \quad .$$

Suppose that $b^2 - 4c \geqslant 0$. Then, since the directrices are identically constant, the solutions of (1) can be categorized as in Figure 2. It is interesting to observe that $y'(x)$ does not depend upon x. Thus, for any point (x_o,y_o), the unique solution that passes through the point (x_o,y_o) must have the same slope as the unique solution that passes through the point (x,y_o) for any x. That is, given a solution of (10), any other solution of (10) in the same region can be realized as $y_o(x-x_o)$ for some x_o. A "standard" curve can thus be constructed for each of the three regions, whose shape is determined only by the constants b and c in (6). This "standard" curve or "template" can be translated along the x-axis to represent the unique solution of (10) passing through any particular point in the region of definition of the "standard" curve. (See Figure 6.)

2. <u>Verification of "Template" Solutions</u>

The closed form solution of (10) can be found, by direct integration of (10), to be

$$y(x) = \frac{r_1 - r_2 C u(\bar{x})}{1 - C u(\bar{x})} \tag{11}$$

where

$$u(\bar{x}) = \exp[(r_1 - r_2)\bar{x}] \quad ,$$

$$\bar{x} = x - x_o \quad ,$$

and
$$C = \frac{y(x_0) - r_1}{y(x_0) - r_2} .$$

Notice that when $Cu(\bar{x}) = 1$, $y(x)$ is infinite. That is, $y(x)$ is finite when

$$\exp[(r_1 - r_2)\bar{x}] = \frac{1}{C} .$$

For a solution $y(x) > r_1$, $0 < C < 1$, so that $1 < 1/C < \infty$. Since $r_1 - r_2 > 0$, there is an $\hat{x}$ such that $\exp[(r_1 - r_2)\hat{x}] = \frac{1}{C}$, so that $y(x)$ is assymptotic to $\bar{x} = \hat{x}$ as $\bar{x} \to \hat{x}^-$. Similarly, for a solution $y(x)$ $< r_2$, $1 < C < \infty$, so that $0 < \frac{1}{C} < 1$. In this case $\bar{x} > 0$ implies that $\exp((r_1 - r_2)\bar{x}) > 1 > \frac{1}{C}$. Thus, $y(x)$ is not infinite for any $\bar{x} > 0$. However, there is an $\hat{x} < 0$, such that $\bar{x} = \hat{x}$ implies that $\exp((r_1 - r_2)\bar{x}) = \frac{1}{C}$. That is, as $x \to \hat{x}^+$, $y(x)$ becomes asymptotic to $\bar{x} = \hat{x}$.

Furthermore, for a solution $y(x)$ such that $r_2 < y(x) < r_1$, a simple transformation of variables reduces (11) into a familiar form. Since any one solution in this region is representative of all of them, let $y(x_o) = (r_1 + r_2)/2$. Let $\bar{y}(\bar{x}) = y(\bar{x}) - (r_1 + r_2)/2$ where $\bar{x} = x - x_o$. Then

$$\bar{y}(\bar{x}) = -\frac{a}{2} \tanh\left(\frac{a}{2}\bar{x}\right) \qquad (12)$$

where $a = r_1 - r_2$.

The "standard" curves or "templates" can be expressed for the three distinct regions as special forms of Equation (11). For example, letting $C = 1$, the "standard" curves obtained are,

$$y(x) = \frac{r_1 - r_2 \exp(ax)}{1 - \exp(ax)}$$

where $a = r_1 - r_2$

for $\quad y(x) \notin [r_1, r_2]$,

and

$$y(x) = -\frac{a}{2}\tanh\left(\frac{a}{2}x\right)$$

where $a = r_1 - r_2$

for $\quad r_2 < y(x) < r_1 \quad .$

These "standard" solutions are exactly those represented in Fig. 6.

Consider now the possibility that $b^2 - 4c < 0$. That is, suppose that $y' = (y-r_1)(y-r_2)$ with $r_1 = c + di$ and $r_2 = c - di$. Direct integration yields

$$y(x) = d\tan[d(x+K)] + c \qquad (13)$$

where K is an integration constant. Consider equation (12) with $a = r_1 - r_2 = 2di$. Then (12) becomes

$$\begin{aligned} y(x) - \frac{a}{2} &= -di\tanh[di(x-x_0)] \\ &= -d\tan[-d(x-x_0)] \end{aligned}$$

Finally, $\quad y(x) = d\tan(dx - dx_0) + \frac{a}{2} \quad . \qquad (13')$

Note that (13') agrees with (13) with $-K = x_0$ and $c = \frac{a}{2}$. Equation (13) is graphed in Fig. 7 for $K = 0$. Note that (13) resembles a 90° counterclockwise rotation of (12).

For the case where $b^2 - 4c \geq 0$, then, the directrix approach gave insight into certain properties of solutions of (10). Specifically, boundedness could be discussed in terms of the region in which the solution appeared. These regions were determined by the directrices. The independence of y' with respect to x suggested a "template" representation of the solutions of (10). The directrix approach, therefore, can give strong geometrical insight to the solutions of (10). What insight can this approach give to Burgers' Riccati equation?

III. APPLICATION OF DIRECTRIX APPROACH TO BURGERS' RICCATI EQUATION

A. CASE I: $\beta = 0$

Now, consider the Burgers'-Riccati equation

$$y' = y^2 - 2\alpha x y + \beta \quad ; \quad \alpha > 0 \tag{14}$$

from the directrix point of view. The case where $\beta = 0$ will be considered first.

Consider the equation

$$y' = y^2 - 2\alpha x y \quad . \tag{15}$$

An equivalent form of (15) is

$$y' = y(y - 2\alpha x) \quad . \tag{16}$$

Note that $y = 0$ satisfies (16) so that no non-zero solution of (16) can cross the x-axis. Hence, only solutions $y(x)$ such that $y(x) > 0$ for all x are considered in the following arguments.

The directrices for (16) are defined by

$$u(x) = \begin{cases} 2\alpha x & \text{for } x > 0 \\ 0 & \text{for } x \leqslant 0 \end{cases}$$

and

$$\ell(x) = \begin{cases} 0 & \text{for } x \geqslant 0 \\ 2\alpha x & \text{for } x < 0 \end{cases}$$

For $y(x) > 0$ a solution of (16), $y'(x)$ is described as follows:

$$y'(x) \begin{cases} = 0 & \text{for} \quad y = 0 \quad \text{or} \quad y = 2\alpha x \\ < 0 & \text{for} \quad 0 < y(x) < 2\alpha x \\ > 0 & \text{for} \quad y(x) > 2\alpha x \end{cases}$$

For r a real number, define

$$\mathcal{C}_r = \left\{ (x,y) \mid y > 0 \text{ and } y^2 - 2\alpha x y - r = 0 \right\} .$$

Then, the solution y(x) of (16) that passes through $(\bar{x},\bar{y}) \in \mathcal{C}_r$, must have slope equal to r when $x = \bar{x}$. Notice that the equation

$$y^2 - 2\alpha x y - r = 0$$

can be written as

$$[y - w(x)][y - v(x)] = 0$$

where

$$w(x) = \alpha x + (\alpha^2 x^2 + r)^{\frac{1}{2}} \tag{17}$$

and

$$v(x) = \alpha x - (\alpha^2 x^2 + r)^{\frac{1}{2}} .$$

In particular, $\mathcal{C}_0 = 2\alpha x$. The family of curves defined by the relation $\mathcal{C}_r$ as r ranges over the real numbers is represented in Figure 8. Notice that for $r > 0$, $\mathcal{C}_r$ defines a single-valued function for positive values of y. For $r > 0$ let $\mathcal{C}_r(x)$ denote that function. $\mathcal{C}_r(x)$ is in fact equal to u(x) for positive y and $r > 0$. The slope of $\mathcal{C}_r(x)$ is given by:

$$\mathcal{C}_r'(x) = \alpha + \frac{\alpha^2 x}{(\alpha^2 x^2 + r)^{\frac{1}{2}}} . \tag{18}$$

Note that $0 < \mathcal{B}'_r(x) < 2\alpha$ for all x, and that $\mathcal{B}'_r(0) =$ for all $r > 0$.

Also, $\mathcal{B}''_r(x)$, given by

$$\mathcal{B}''_r(x) = \frac{\alpha(\alpha^2x^2 + r) + \alpha^3x^2}{(\alpha^2x^2 + r)^{\frac{3}{2}}},$$

is positive for all x.

Suppose y(x) is a solution of (16) such that for some $x_o > 0$, $y(x_o) < 2\alpha x_o$. Then $y'(x_o) < 0$, and as $x \to \infty$, $y(x) \to 0$; that is, y(x) is bounded as $x \to \infty$. Thus, any solution y(x) such that $y(x_o) < 2\alpha x_o$ for some $x_o > 0$ is bounded as $x \to \infty$. Moreover, since no bounded solution can be entirely above $y = 2\alpha x$ for all $x > 0$, then every bounded solution must lie between the directrices for all $x > x_o$ for some x_o. That is, the existence of an x_o such that $x > x_o$ implies that $y(x) \to 0$ as $x \to \infty$ completely characterizes bounded solutions of (16).

Now, suppose that for some $x_o > 0$, $2\alpha x_o < y(x_o) \leq \mathcal{B}_\alpha(x_o)$. Then $0 < y'(x_o) \leq \alpha$. From equation (16), y''(x) is found to be

$$y''(x) = 2[y(y' - \alpha) - \alpha x y'] \tag{19}$$

Thus, $y''(x_o) < 0$ since $y' - \alpha \leq 0$ and $-\alpha xy' < 0$. Hence, y(x) must intersect $2\alpha x$ for some $\bar{x} > 0$. By a previous argument y(x) must then be bounded as $x \to \infty$.

Suppose that for some $\bar{x} > 0$, $y(x) \geq \mathcal{B}_{2\alpha}(x)$. Then $y'(x) \geq 2\alpha$. Since $\alpha < \mathcal{B}'_r(x) < 2\alpha$, y'(x) must be increasing as x increases from $\bar{x}$. Alternately, from (19),

$$\begin{aligned} y''(\bar{x}) &\geq 2[y(y' - \alpha) - \tfrac{y}{2}y'] \\ &\geq 2[y(\tfrac{y'}{2} - \alpha)] \geq 0 \end{aligned}.$$

Thus, y(x) cannot intersect y = 2 x and is unbounded as $x \to \infty$.

To summarize the results (See Figure 9):

(1) If for some $x_o > 0$, $y(x_o) \leq \mathcal{C}_\alpha(x_o)$, then $y(x)$ is bounded as $x \to \infty$.

(2) If for some $\bar{x} > 0$, $y(\bar{x}) \geq \mathcal{C}_{2\alpha}(\bar{x})$, then $y(x)$ is unbounded as $x \to \infty$.

B. CASE II: $\beta = 2\alpha$

It is interesting to notice that the solutions of Burgers' Riccati equation are symmetric with respect to the origin. That is,

$$\frac{d(-y)}{d(-x)} = (-y)^2 - 2\alpha(-x)(-y) + \beta = y^2 - 2\alpha x y + \beta = \frac{dy}{dx}$$

It is thus sufficient to consider only those solutions in the half-plane where $x \geqslant 0$. There is, by symmetry, a solution in this half-plane which corresponds to any solution entirely contained in the half-plane determined by $x < 0$. This fact is helpful in considering Equation (14) with $\beta \neq 0$.

Equation (14) will now be considered for $\beta = 2\alpha$, that is

$$y' = y^2 - 2\alpha x y + 2\alpha \quad . \qquad (20)$$

As in the case where $\beta = 0$, define a family of curves:

$$\mathcal{B}_r = \left\{ (x,y) \mid y^2 - 2\alpha x y + 2\alpha - r = 0 \right\}$$

Notice that for $y > 0$, $\mathcal{B}_r = {}_{r-2}$. In fact, for $y > 0$, the family of curves $\mathcal{B}_r$ is merely the family of curves $\mathcal{C}_r$ with the subscripts increased by 2α. In particular, the directrices of (20) are actually $\mathcal{B}_o = \mathcal{C}_{-2\alpha}$. (See Figure 9.)

Notice that $y = 0$ does not satisfy (20). Thus, solutions may cross the x-axis, and in fact, cross with slope equal to 2α. However, $y = 2\alpha x$ does satisfy (20) so that no solution can cross the line $y = 2\alpha x$. Notice

also that $y = 2\alpha x$ is precisely $\mathcal{B}_{2\alpha}(x)$, since the solution $y = 2\alpha x$ has slope equal to 2α for all x. The following discussion pertains to all solutions y(x) of (20) for which $y(x) > 0$ for all x.

Suppose that for some $\bar{x} > 0$, $y(\bar{x}) = 0$. Then for some $x' > \bar{x}$, $y'(x') = 0$; that is, $y(x') \in \mathcal{B}_o$. If not, then $y \geq 2\alpha x$ for $x \geq \bar{x}$, which contradicts that $y(\bar{x}) = 0$. Thus, since $x > x'$ forces $y'(x) < 0$, y(x) must be bounded as $x \to \infty$. Notice also that if $y(\hat{x}) = 0$ for $\hat{x} < 0$, then $y'(x) > 2\alpha$ for $x > \hat{x}$. Since for $x > \hat{x}$, $y(x) > 2\alpha x$, then y(x) must be unbounded as $x \to \infty$.

The boundedness of a solution y(x) of (20) as $x \to \infty$ can thus be determined by the sign of the x-intercept of y(x). Notice that every solution y(x) has an x-intercept so that this characterization for boundedness of solutions is meaningful. Moreover, note that if y(x) is bounded as $x \to \infty$, it is unbounded as $x \to -\infty$, and vice-versa (that is, interchanging ∞ and $-\infty$), except for the case $y = 2\alpha x$. Furthermore, the characterization for bounded solutions could also be given in terms of the y-intercept. That is, $y(0) > 0$ implies that y(x) must have a negative x-intercept, so that $y(0) > 0$ leads to an unbounded solution as $x \to \infty$. Similarly, $y(0) < 0$ leads to a bounded solution as $x \to \infty$.

To summarize the results (See Figure 11):

(1) If $y(0) \leq 0$, y(x) is bounded as $x \to \infty$.

(2) If $y(0) > 0$, y(x) is unbounded as $x \to \infty$.

C. CASE III: $\beta > 0$

If were allowed to take on all values between 2α and zero, that point p on the y-axis corresponding to $\beta = 2\alpha$, initial values above

which cause the solution to be unbounded and below which cause the solution to be bounded as $x \to \infty$, should take on a continuum of values between zero and the corresponding point q for $\beta = 0$. It is known that q lies in the interval $(\sqrt{\alpha}, \sqrt{2\alpha})$, but its exact value is not known through this geometric approach. As β takes on values above 2α, p moves down the y-axis until eventually $p = -\infty$.

Note that increasing β causes a shift in the labeling of the family of curves $\mathcal{C}_r(x)$ for $r \in \mathbb{R}$. In particular, the lines $y = 2\alpha x$ and $y = 0$ are always $\mathcal{C}_\beta(x)$ for any β. It should be expected, then, for some value of $\beta > 2\alpha$, that every solution which passes through the y-axis will be unbounded. The particular symmetry properties of the solutions, at least in the present case, suggests that if a solution is bounded for $x \to \infty$ and $\beta > 2\alpha$, then it is unbounded as $x \to -\infty$, This statement holds, of course, for $\beta = 2\alpha$, since if $y(x)$ is bounded as $x \to \infty$, then $y(0) < 0$. Thus $-y(0) > 0$, which implies that $-y(x)$ is unbounded as $x \to -\infty$. For $\beta > 2\alpha$, $p < 0$, so that for any bounded solution $y(x)$ as $x \to \infty$, $y(0) < p$. Thus, $-y(0) > -p$, so that $-y(x)$ must be unbounded as $x \to -\infty$. Note that the converse of the result is not true in general. That is, some solutions are unbounded both as $x \to \infty$ and also as $x \to -\infty$. A simple example of this for $\beta = 2\alpha$ is the particular solution $y = 2\alpha x$.

IV. CLOSED FORM SOLUTIONS

A. CASE I: $\beta = 0$

The closed form solution to Burgers' Riccati equation can be obtained with certain restrictions on α and β .

Suppose $\beta = 0$. That is, a solution of

$$y' = y^2 - 2\alpha x y \tag{15}$$

is desired. The closed form solution is known (see Kamke [Ref. 6] and can be obtained from

$$\frac{1}{y(x)} = \exp\left[\int_0^x 2\alpha t\,dt\right] \int_0^x \exp\left[-\int_0^t 2\alpha s\,ds\right] dt$$

Thus,

$$y(x) = -\frac{\exp\left[-\int_0^x 2\alpha t\,dt\right]}{\int_0^x \exp\left[-\int_0^t 2\alpha s\,ds\right] dt} \tag{21}$$

B. CASE II: $\beta = 2\alpha$

Suppose that $\beta = 2\alpha$. It was observed that $y = 2\alpha x$ is a particular solution of

$$y' = y^2 - 2\alpha x y + 2\alpha \quad . \tag{20}$$

Let $y = \frac{1}{u} + 2\alpha x$. Then, substituting into (16),

$$u' - 2\alpha x u - 1 = 0 \quad .$$

The solution of this non-homogeneous linear first order equation is given by

$$u(x) = \exp\left[-\int_0^x 2\alpha t\,dt\right]\left[u(0) - \int_0^x \exp\left[\int_0^t 2\alpha s\,ds\right] dt\right].$$

Since $$y = \frac{1}{u} + 2\alpha x \quad ,$$

$$y(x) = \frac{\exp[\int_0^x 2\alpha t\,dt]}{u(0) - \int_0^x \exp[\int_0^t 2\alpha s\,ds]\,dt} + 2\alpha x \ . \tag{22}$$

Equation (22) is the closed form solution of (20).

For other values of β, the solutions of Burgers' Riccati equation are not as easily found. For certain classes of β, however, solutions can be found in terms of known functions. These solutions will be developed in the following pages.

C. METHOD OF FROBENIUS

It is well known that the Riccati equation is equivalent to the linear second order differential equation. That is, to each Riccati equation there corresponds a linear differential equation of second order, and vice-versa. In seeking solutions to

$$y' = y^2 - 2\alpha x y + \beta \tag{14}$$

it is advantageous to study the second order linear form of (14). If, in particular, $y = -u'/u$, then (14) becomes

$$u'' + 2\alpha x u' + \beta u = 0 \quad . \tag{23}$$

The method of Frobenius (e.g., Wylie [Ref. 7]0 can be applied to (23). For example, suppose that

$$u = x^r [a_0 + a_1 x + a_2 x^2 + \cdots] \tag{24}$$

Substituting for u back in (23) and equating coefficients of like powers of x, the coefficients a_i, i = 1,2,3,..., can be found. The roots of the indicial equation turn out to be r = 0 and r = 1. Since the two roots differ by an integer, only one series solution can result. Thus, r = 0 is chosen for simplicity. The values a_o and a_1 are arbitrary constants, and the values of the other coefficients are

$$a_{2K+2} = \frac{(-1)^{K+1} a_o[2\alpha(2K)+\beta][2\alpha(2K-2)+\beta]\cdots[\beta]}{(2K+1)!}$$

for K = 0,1,2,3,... and

$$a_{2K+1} = \frac{(-1)^{K} a_1[2\alpha(2K-1)+\beta][2\alpha(2K-3)+\beta]\cdots[2\alpha+\beta]}{(2K+1)!}$$

for K = 1,2,3,...

Thus, one solution of (23) is (24) with the a_i's defined as above. To obtain a second solution, assume that $u(x) = \phi(x)\, u_1(x)$, where $u_1(x)$ is the solution of (23) corresponding to (24) with r = 0. Substituting for u(x) in (23), and noting that $u_1(x)$ satisfies (23), $\phi(x)$ must then satisfy

$$u_1\phi'' + (2u_1' + 2\alpha x u_1)\phi' = 0 .$$

Solving for ϕ,

$$\phi(x) = C\int_0^x \frac{\exp(-\alpha t^2)}{u_1^2(t)}\,dt + K .$$

Then, $u = \phi u_1$, so that the solution of (14) $y = -(u')/u$ becomes,

$$y(x) = -\frac{u_1'(x)}{u_1(x)} - \frac{C\exp(-\alpha x^2)}{u_1^2(x)\left[K + C\int_0^x \frac{\exp(-\alpha t^2)}{u_1^2(t)}\,dt\right]} . \qquad (25)$$

Thus, (25) is the general solution of (14).

It is interesting to note that for $\beta = 0$, all of the even indexed coefficients of (24) become zero. If a_1 is taken to be zero, $u_1 \equiv a_0$. Thus, (25) reduces to

$$y(x) = \frac{-\exp(-\alpha x^2)}{(K/a_0^2 C) + \int_0^x \exp(-\alpha t^2)dt}$$

which agrees with (21) if the integration constant K is taken to be zero.

Similarly, for $\beta = 2\alpha$, if a_1 is taken to be zero, $u_1 = \exp(-\alpha x^2)$. Substituting for u_1 in (25) gives

$$y(x) = 2\alpha x + \frac{\exp(\alpha x^2)}{(-K/C) - \int_0^x \exp(\alpha t^2)dt} ,$$

which agrees with (22) where $u(0) = -\frac{K}{c}$.

D. ABDELKADER'S METHOD

The closed form solution can be obtained for an infinite class of β's. The following method is due to Abdelkader [Ref. 8]. Consider

$$y' = y^2 - 2\alpha x y + \beta \quad . \tag{14}$$

Let $y(x) = a(x) - v'(x)/v(x)$. Substituting into (10) for y,

$$v'' - [2a - 2\alpha x]v' + [\beta + \alpha(-2\alpha x) + 4\alpha^2 x^2 - a']v = 0,$$

or,

$$v'' - A(x)v' + B(x)v = 0 \quad . \tag{26}$$

If $a(x)$ can be chosen so that $A'(x) = B(x)$, then (26) can be solved easily for $v(x)$. That is, (26) becomes

$$v'' - [A(x)v]' = 0$$

from which

$$v' = A(x)v + C_1$$

can be solved easily for $v(x)$. Then the general solution of (14) can be written

$$y(x) = a(x) - A(x) - \frac{C_1}{v(x)} \quad .$$

The function a(x) is called the "conjugate" to y(x) and satisfies

$$a' = -a^2 + 2\alpha x a - (\beta - 2\alpha) \quad . \tag{27}$$

If $\beta = 2\alpha$, (27) can be solved easily for a(x) (see Kamke [Ref. 6]). If $\beta = 0$, then $a(x) = 2\alpha x$ is a particular solution of (27), which is all that is needed.

If (27) cannot be solved easily as is the case if $\beta = 0$ or $\beta = 2\alpha$, then to repeat the above procedure on equation (27) would merely give y(x) as the "conjugate" to a(x). The transformation $w = 1/a$ yields a distinct Riccati equation, namely

$$w' = 1 - 2\alpha x w + (\beta - 2\alpha) w^2 \quad . \tag{28}$$

Let $w = a_1 - u'/u$ so that (28) goes into

$$u'' - [-2\alpha x + 2a_1(\beta - 2\alpha)]u' + (\beta - 2\alpha)[1 + a_1(-2\alpha x) + a_1^2(\beta - 2\alpha) - a_1']u = 0$$

or

$$u'' - A_1(x)u' + B_1(x)u = 0 \quad .$$

If $a_1(x)$ can be chosen so that $A'(x) = B(x)$, then u(x) can be found as v(x) was found in (26). Then $a_1(x)$, the "conjugate" of (x), must satisfy

$$a_1' = -(\beta - 2\alpha)a_1^2 + 2\alpha x a_1 - \frac{(\beta - 4\alpha)}{(\beta - 2\alpha)} \quad . \tag{29}$$

Notice that if $\beta = 4\alpha$, (29) can be easily solved and $\omega(x)$ can be written

$$\omega(x) = a_1(x) - A_1(x) - \frac{C_2}{u(x)} \quad .$$

Tracing back, y(x) can be found. Note that C_2 can be taken to be zero, since only particular solutions are needed at all except the last step.

It is interesting to note the form of succeeding conjugates $a_i(x)$. For example, $a_2(x)$ turns out to satisfy the equation

$$a_2' = -\frac{(\beta-4\alpha)}{(\beta-2\alpha)} a_2^2 + 2\alpha x a_2 - \frac{(\beta-6\alpha)(\beta-2\alpha)}{(\beta-4\alpha)} \quad .$$

In general, if $a_i(x)$ satisfies

$$a_i' = -A a_i^2 + 2\alpha x a_i - B \quad ,$$

then $a_{i+1}(x)$ satisfies

$$a_{i+1}' = -B\, a_{i+1}^2 + 2\alpha x a_{i+1} - \frac{C}{B}$$

where $C = (\beta - 2(i+1)\alpha)$. Thus, if $\beta = 2n\alpha$, then the (n+1)st conjugate, $a_n(x)$, can be solved easily. Working backwards, y(x) is eventually obtained.

Thus, this infinite class of β's leads to a closed form solution of (14) and is defined by $\beta = 2n\alpha$ for n a natural number.

E. HERMITE POLYNOMIAL SOLUTIONS

Consider, now, the equation

$$y' = y^2 - 2\alpha x y + \beta \tag{14}$$

and apply the transformation $y = \sqrt{\beta} / \omega(t)$, $t = \sqrt{\beta}\, x$.

Then, (10) becomes

$$\frac{d\omega}{dt} = \omega^2 + 2\gamma t \omega + 1 \quad , \quad \text{where} \quad \gamma = \frac{\alpha}{\beta} \quad .$$

Thus, the number of parameters has been reduced from two to one. Let $\omega = -V'/V$ to obtain the second order form

$$V'' - 2\gamma t V' + V = 0 \quad , \qquad \text{where} \quad ' = \frac{d}{dt} \quad .$$

Now, letting $t = \sqrt{\gamma}\,\xi$

$$\gamma V'' - 2\gamma\xi V' + V = 0$$

where differentiation now is with respect to ξ . Dividing by γ ,

$$V'' - 2\xi V' + \frac{1}{\gamma} V = 0 \quad . \tag{30}$$

Now observe that the equation

$$y'' - 2xy' + 2ny = 0 \quad , \quad \text{for n a natural number,}$$

has $H_n(x)$, the Hermite polynomial order n, as a particular solution. Thus, for $(\gamma)^{-1} = 2n$ for n a natural number, equation (30) has $V(\xi) = Hn(\xi)$ as a particular solution. The condition $(\gamma)^{-1} = 2n$ can be re-written as $\beta/\alpha = 2n$ or $\beta = 2n\alpha$. Thus, when $\beta = 2n\alpha$, for n a natural number, equation (14) has a solution in terms of Hermite polynomials. Notice that the condition that $\beta = 2n\alpha$ is precisely the restriction upon the class of solvable Burgers' equations using Abdelkader's method.

F. HYPERGEOMETRIC FUNCTION SOLUTIONS

Still another approach to Burgers' equation comes from its linear second order form. Letting $y = -u'/u$,

$$u'' + 2\alpha x u' + \beta u = 0 \quad . \tag{31}$$

Equation (31) has the solution (see Kamke, [Ref. 6], and [Ref. 9]).

$$u(x) = x^{-\frac{1}{2}} \exp\left(-\frac{\alpha x^2}{2}\right) M\left(1-\frac{\beta}{4\alpha}, \frac{3}{2}, \alpha x^2\right)$$

where M(a,b,x), the Kummer's Function, satisfies

$$x\omega'' + (b-x)\omega' - a\omega = 0 \quad .$$

Burgers' equation thus has solutions in terms of confluent bypergeometric functions.

V. CONCLUSIONS

A. BOUNDEDNESS PROPERTIES FOR $\beta = 2n\alpha$

It is interesting to note that if, in Burgers' equation, $\beta = 2n\alpha$, then a particular solution, y_1, is known in terms of $H_n(x)$. Letting $y = y_1 + 1/v$ in (10), v must satisfy

$$v' + (-2\alpha x + 2y_1)v = -1 .$$

That is,

$$v(x) = \frac{C - \int_0^x A(t)dt}{A(x)}$$

where

$$A(x) = \exp\left[\int_0^x (-2\alpha t + 2y_1(t))dt\right] .$$

Then the solution of (14) becomes

$$y = y_1 + \frac{A(x)}{C - \int_0^x A(t)dt} \tag{32}$$

From (32), $y(0) = y_1(0) + \frac{1}{C}$ and

$$C = (y(0) - y_1(0))^{-1} .$$

Note that $y_1(0)$ is known. Also note from (32) that y(x) becomes unbounded when

$$C = \int_0^x A(t)dt .$$

That is, given y(0) the constant C is completely specified. Then the solution y(x) corresponding to y(0) will become infinite for that value of x for which

$$\frac{1}{y(0) - y_1(0)} = \int_0^x A(t)dt . \tag{33}$$

Therefore, the evaluation of the integral $\int_0^x A(t)dt$ will yield the value of x at which an unbounded solution has a vertical assymptote. If no such value x exists as $x \to \infty$, then y(x) must be a bounded solution.

Note that formula (33) is valid for any particular solution of (14). In particular, for $\beta = 2n\alpha$, formula (33) will tell if y(x) is unbounded for a particular y(0) and in fact will give the assymptotic value for y(x) in the case that y(x) is unbounded.

In this way the geometric directrix approach to the Riccati equation is supplemented. That approach, it will be recalled, is applicable whenever a moving point has a pair of directrix points above, below or around it to guide it. However, the directrix will not be present if the roots of the factored Riccati are complex. In this case the solution becomes unbounded. It is precisely in this case, that the above information will be useful.

B. VERIFICATION OF GEOMETRICAL OBSERVATIONS

The geometrical observations made for the cases $\beta = 0$ and $\beta = 2\alpha$ will now be evaluated in light of formula (33).

For $\beta = 0$, it was observed that $y(0) < \sqrt{\alpha}$ implied that y(x) was bounded as $x \to \infty$. On the other hand $y(0) > \sqrt{2\alpha}$ meant that y(x) was unbounded as $x \to \infty$. Thus, for the $\beta = 0$ case, $y_1(x) \equiv 0$, and $A(x) = \exp(\int_0^x -2\alpha t \, dt)$. From (33),

$$\frac{1}{y(0)} = \int_0^x \exp(-\alpha t^2)dt \quad .$$

Notice that for $s = t/(2\alpha)^{\frac{1}{2}}$,

$$\int_0^x \exp(-\alpha t^2)dt = \frac{1}{\sqrt{2\alpha}} \int_0^{\sqrt{2\alpha}\,x} \exp(-t^2/2)\,dt$$

$$= \sqrt{\frac{\pi}{\alpha}} \left[\frac{1}{\sqrt{2\alpha}} \int_0^{\sqrt{2\alpha}\,x} \exp(-t^2/2)dt . \right.$$

Thus,
$$0 < \int_0^x A(t)\,dt < \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \quad .$$

Hence, y(x) will be unbounded and have a vertical assymptote if and only if $2\sqrt{\pi/\alpha} < y(0) < \infty$; that is, if and only if $\sqrt{\frac{4\alpha}{\pi}} < y(0) < \infty$. Notice that $\sqrt{\alpha} < \sqrt{\frac{4\alpha}{\pi}}$, so that $y(0) < \sqrt{\alpha}$ implies that y(x) has no vertical assymptote as $x \to \infty$, and, hence, must be bounded. Moreover, $\sqrt{\frac{4\alpha}{\pi}} < \sqrt{2\alpha}$, so that $y(0) > \sqrt{2\alpha}$ implies that y(x) has a vertical assymptote. Thus, the analytic expression has produced a sharp bound for the initial values of solutions that are bounded as $x \to \infty$. The geometric insight gained from the directrix approach, however, can be intuitively helpful not only to determine boundedness but also to visualize the general behavior of solutions as the initial value varies. The conclusions drawn from the geometric approach, though not as precise as the analytic in this case, are nevertheless verified.

For $\beta = 2\alpha$ it was observed that any positive y(0) led to an unbounded solution, and that any negative y(0) led to a bounded solution. Here, unlike the case where $\beta = 0$, a sharp bound was obtained from the directrix approach, namely the value y(0) = 0. In this case, $y_1 = 2\alpha x$ and $A(x) = \exp(\int_0^x(-2\alpha t + 4\alpha t)dt) = \exp(\alpha x^2)$. Then, from (33), since $y_1(0) = 0$,
$$\frac{1}{y(0)} = \int_0^x \exp(\alpha t^2)\,dt \quad .$$
Notice that $0 < \int_0^x \exp(\alpha t^2)dt < \infty$. Thus, any positive y(0) will have a vertical assymptote and any negative y(0) will not have vertical assymptote, as $x \to \infty$. Hence, the observations are again valid. Notice that in this case the directrix approach led to the same sharp bound as was produced in the analytic argument.

Figure 1 - Position of $y(x_o)$ relative to $l(x_o)$ and $u(x_o)$

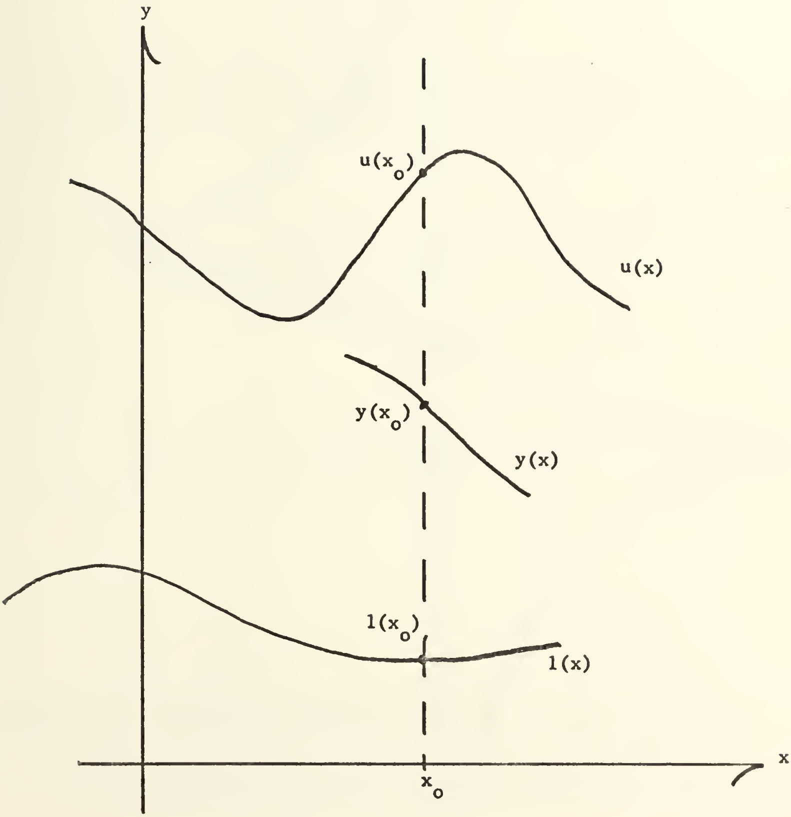


Figure 2 - Constant directrices $u(x) = C_2$; $l(x) = C_1$

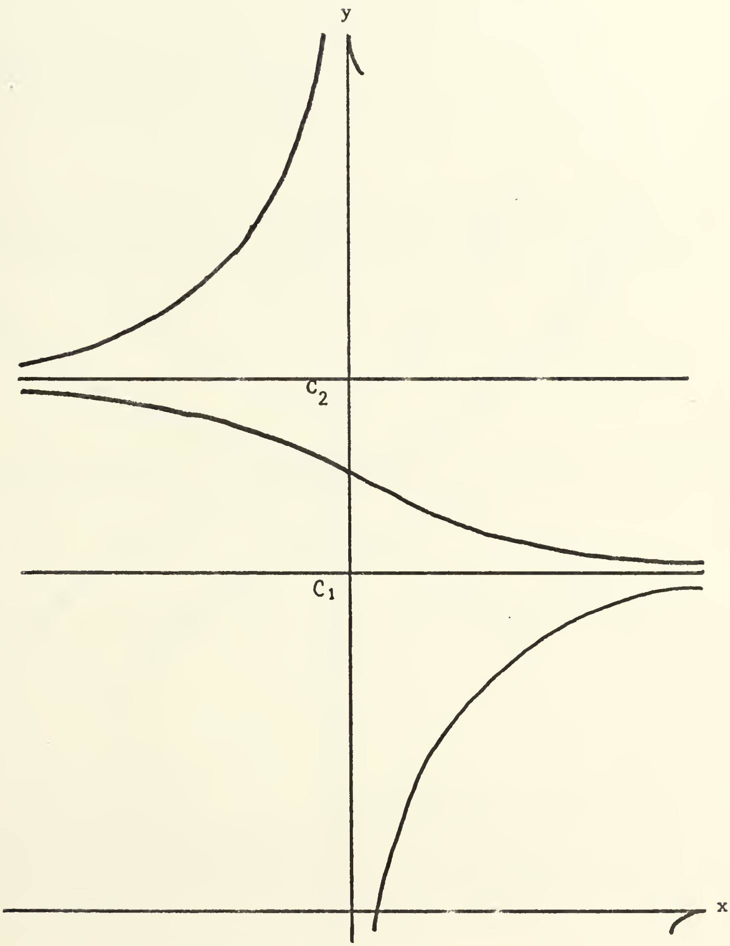


Figure 3 - Riccati "lens"   A = Focusing Region
B = Scattering Region

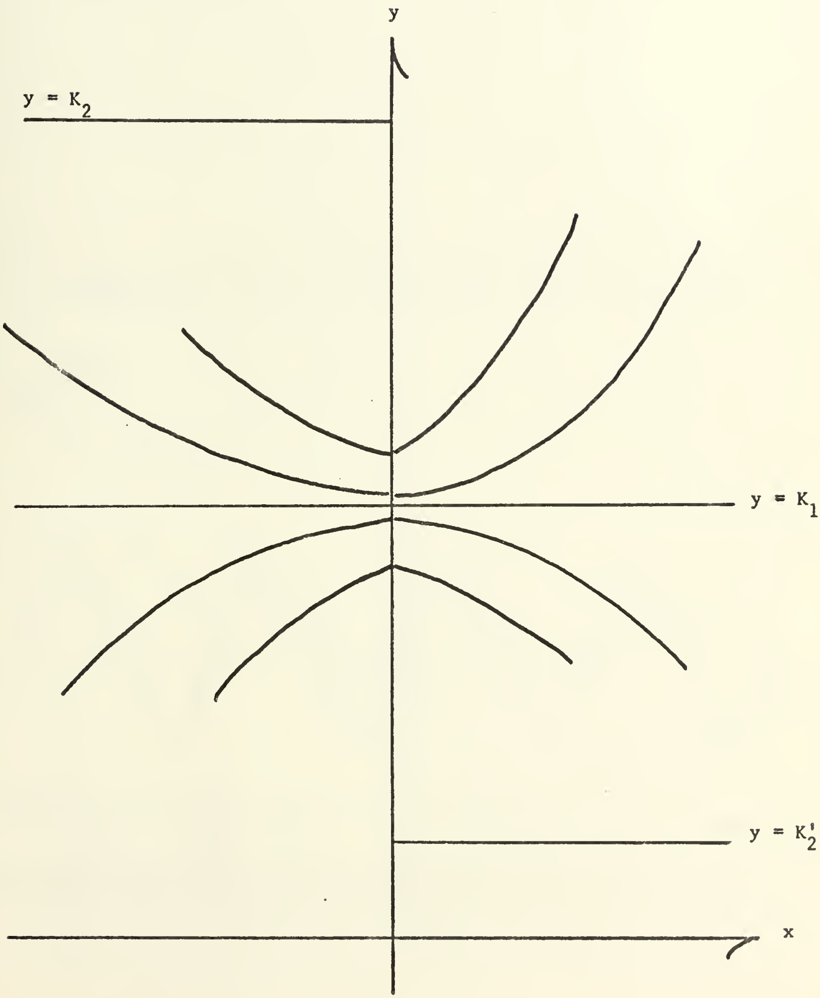


Figure 4 - Oscillating Solution

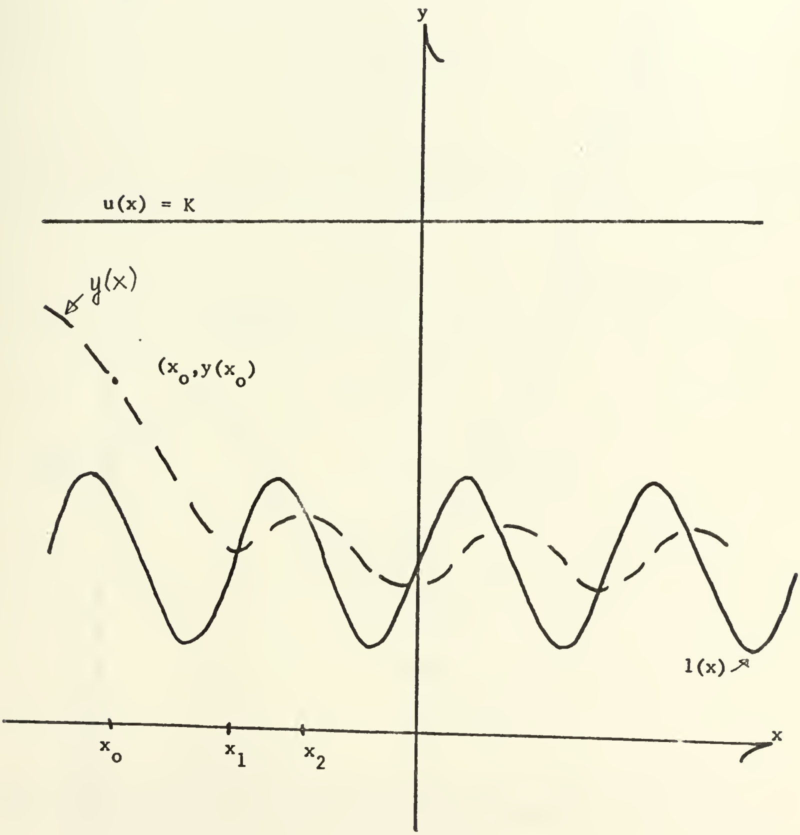


Figure 5 - Random constant directrices

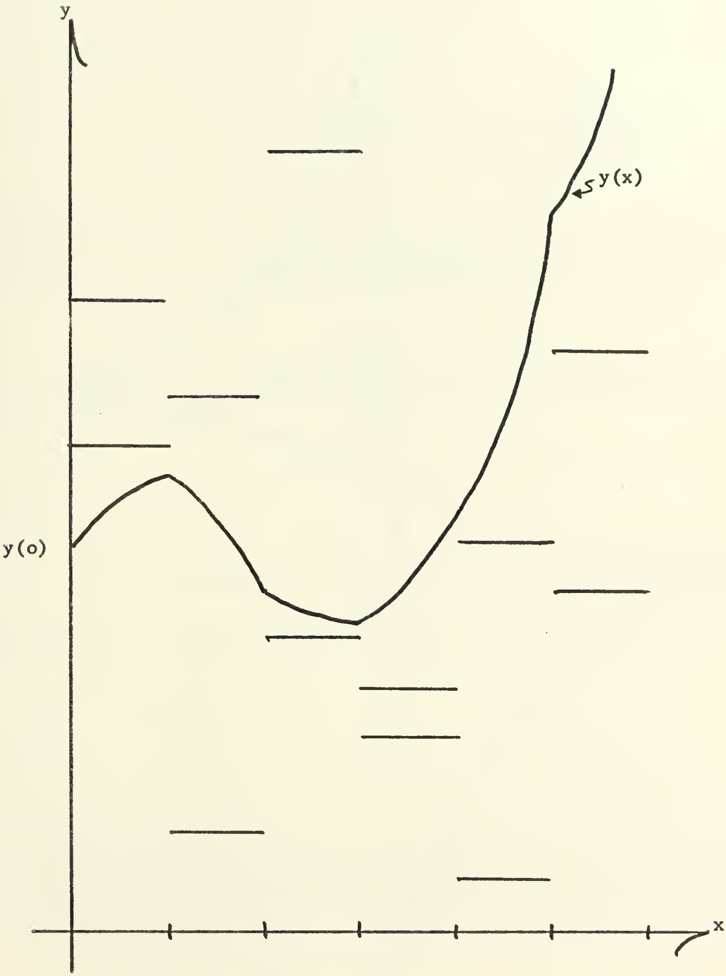


Figure 6 - "Template" solution through (x_0,y_0)

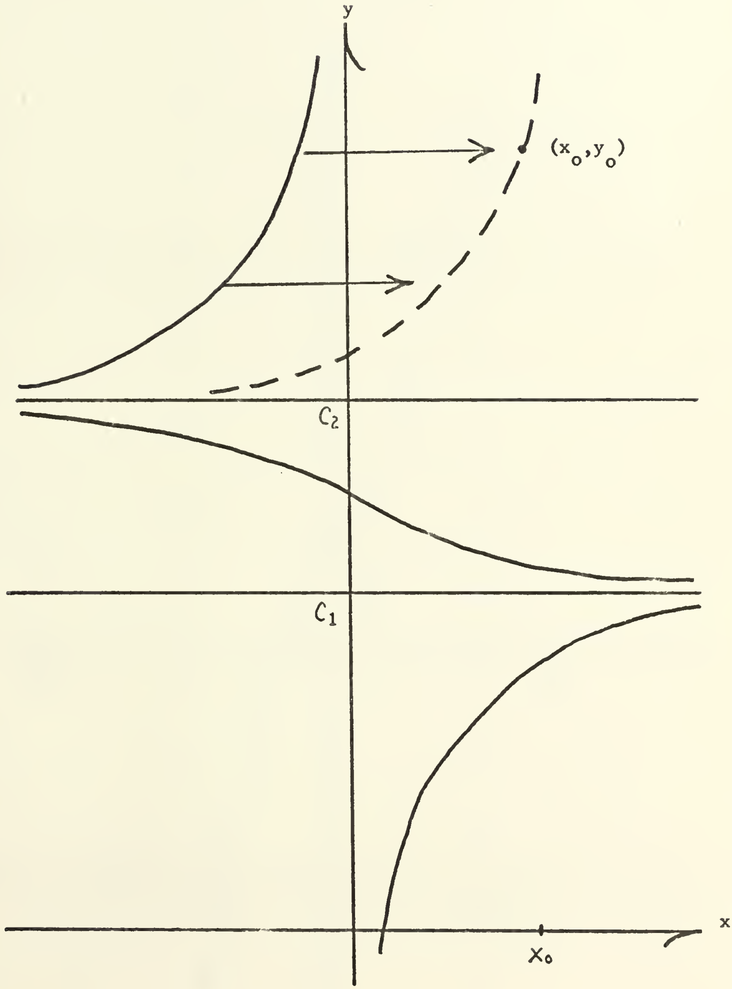


Figure 7 - Complex directrices u(x) = C + di; l(x) = C-di

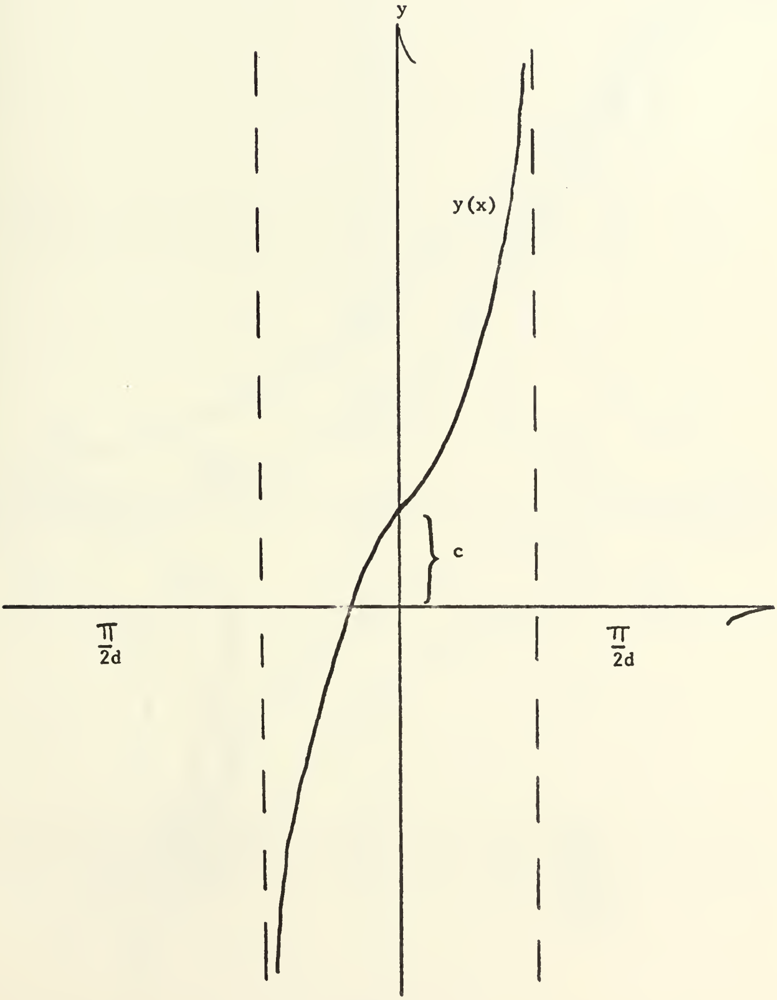


Figure 8 - Family of $\mathcal{C}_r$ curves

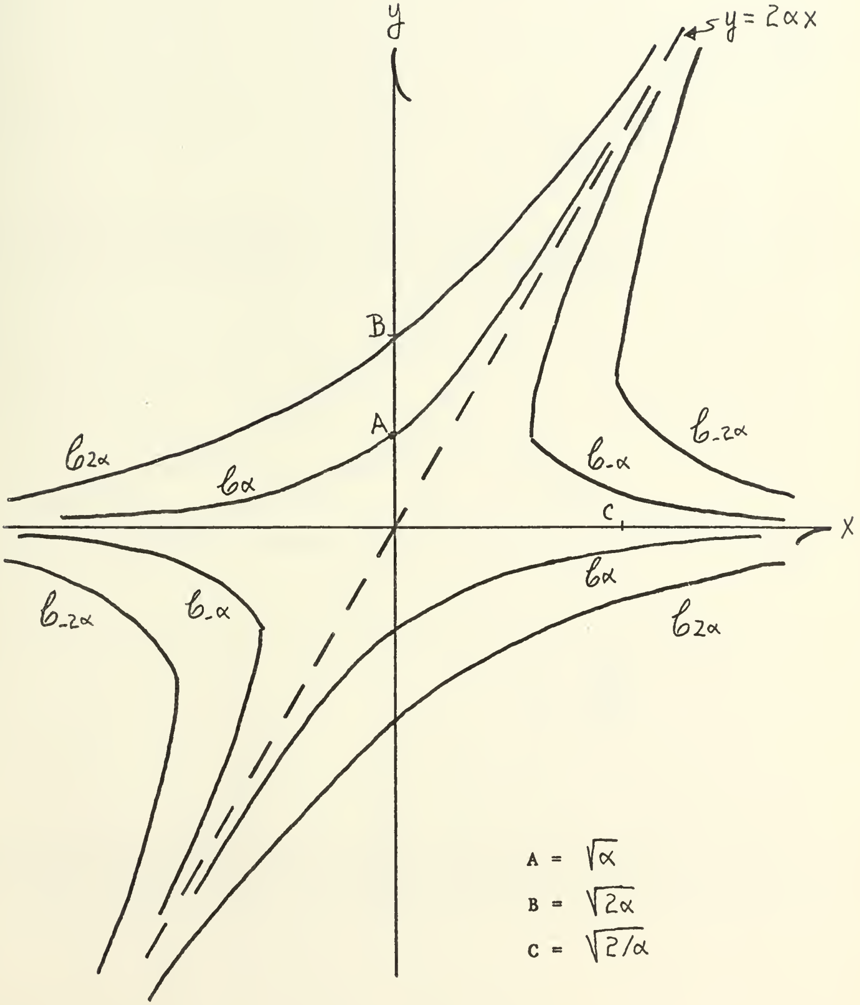
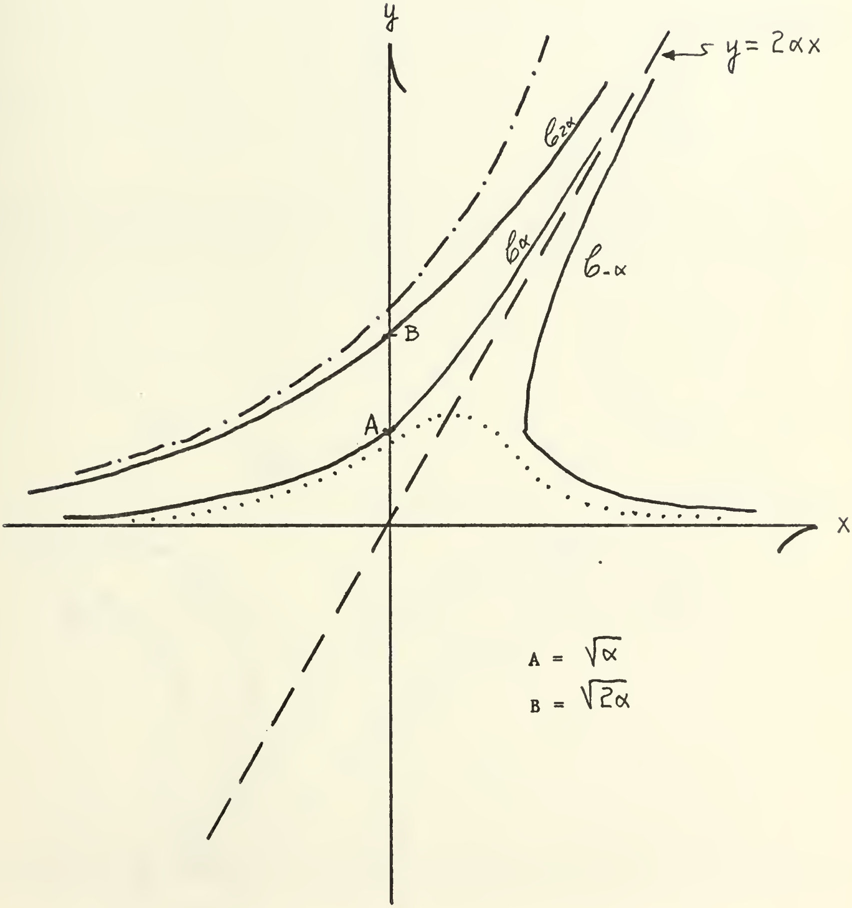


Figure 9 - Examples of bounded and unbounded solutions for

$\beta = 0$



A = $\sqrt{\alpha}$

B = $\sqrt{2\alpha}$

Figure 10 - Family of $\mathcal{B}_r$ curves

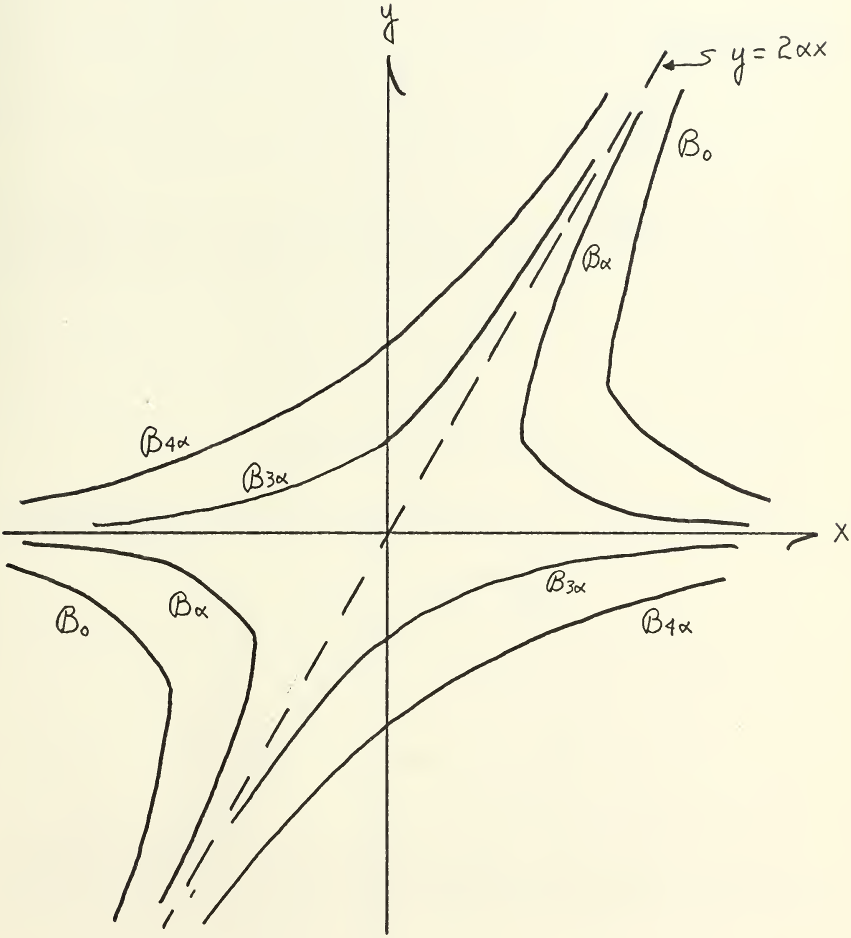
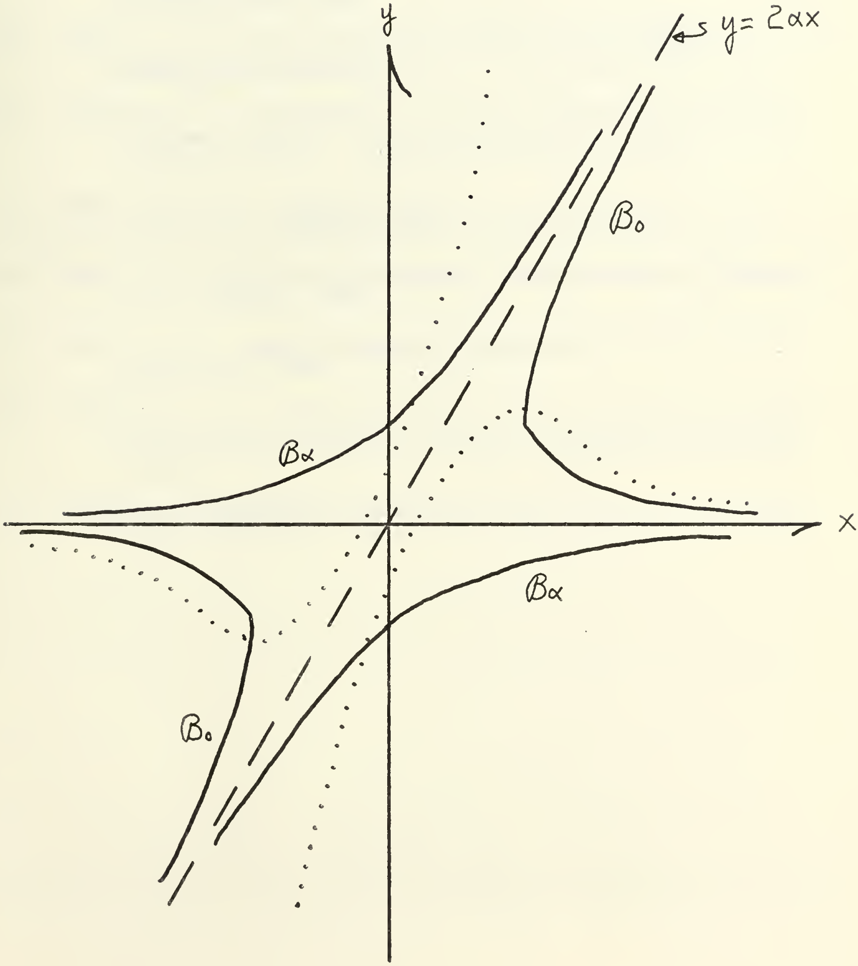


Figure 11 - Examples of bounded and unbounded solutions

as $x \to \infty$ for $\beta = 2\alpha$



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13. ABSTRACT

The directrix approach, which gives powerful insight to the geometric structure of solutions of the general Riccati equation, is developed.

Burgers' Riccati equation is derived, and conclusions are drawn utilizing the directrix approach concerning the boundedness properties of solutions of this equation with certain restrictions on parameter values.

Closed form solutions are developed for Burgers' Riccati equation for certain parameter values. A method is produced to obtain the vertical assumptote for unbounded solutions of Burgers' Riccati equation with certain restrictions on parameter values. Conclusions drawn from the application of the directrix method to Burgers' equation are verified.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Riccati Equation						
Burgers' Equation						
Nonlinear Differential Equation						
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Geometric Approach to General Riccati Equation						
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